UNCLASSIFIED

AD **4 6 4 3 7 3**

DEFENSE DOCUMENTATION CENTER

FOR

SCIENTIFIC AND TECHNICAL INFORMATION

CAMERON STATION ALEXANDRIA, VIRGINIA

UNCLASSIFIED

THE
ANTENNA
LABORATORY

RESEARCH ACTIVITIES in - - -

Automatic Controls Antennas Echo Area Studies
Microwave Circuits Astronautics E M Field Theory
Terrain Investigations Radomes Systems Analysis
Wave Propagation Submillimeter Applications

ON THE OPTIMUM GAIN OF UNIFORMLY
SPACED ARRAYS OF ISOTROPIC
SOURCES OR DIPOLES

by

C. T. Tai

Contract Number N123(953)-31663A
Prepared for
U. S. Navy Purchasing Office
Los Angeles 55, California

1522-1   31 March 1963

Department of ELECTRICAL ENGINEERING

THE OHIO STATE UNIVERSITY
RESEARCH FOUNDATION
Columbus, Ohio

NOTICES

When Government drawings, specifications, or other data are used for any purpose other than in connection with a definitely related Government procurement operation, the United States Government thereby incurs no responsibility nor any obligation whatsoever, and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data, is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use, or sell any patented invention that may in any way be related thereto.

The Government has the right to reproduce, use, and distribute this report for governmental purposes in accordance with the contract under which the report was produced. To protect the proprietary interests of the contractor and to avoid jeopardy of its obligations to the Government, the report may not be released for non-governmental use such as might constitute general publication without the express prior consent of The Ohio State University Research Foundation.

Qualified requesters may obtain copies of this report from the ASTIA Document Service Center, Arlington Hall Station, Arlington 12, Virginia. Department of Defense contractors must be established for ASTIA services, or have their "need-to-know" certified by the cognizant military agency of their project or contract.

R E P O R T

by

THE OHIO STATE UNIVERSITY RESEARCH FOUNDATION
COLUMBUS 12, OHIO

Cooperator

U. S. Navy Purchasing Office
929 South Broadway
Box 5090 Metropolitan Station
Los Angeles 55, California

Contract Number

N123(953)-31663A

Investigation of

Study Program Related to Shipboard Antenna
System Environment

Subject of Report

On the Optimum Gain of Uniformly Spaced
Arrays of Isotropic Sources or Dipoles

Submitted by

C. T. Tai
Antenna Laboratory
Department of Electrical Engineering

Date

31 March 1965

TABLE OF CONTENTS

ABSTRACT

The optimum gain of uniformly spaced arrays of isotropic sources or dipoles is investigated theoretically in this paper. The formulation is processed with the aid of an array matrix. The optimum gain and the corresponding excitation are expressed directly in terms of the elements of the array matrix.

ON THE OPTIMUM GAIN OF UNIFORMLY SPACED
ARRAYS OF ISOTROPIC SOURCES OR DIPOLES

INTRODUCTION

For a uniformly spaced array consisting of a given number of isotropic sources, the renowned synthesis of Dolph,[1] including the extension by Riblet,[2] offers the minimum beamwidth for a prescribed side-lobe level. It is not an optimum design from the point of view of maximizing the directivity or the gain. Ma and Cheng[3] have recently presented another synthesis which optimizes the gain under the condition of a prescribed side-lobe level. Because of the polynomial formulation, neither of these two methods can conveniently be applied to arrays of directive sources, such as those consisting of dipoles. The problem that deals solely with the maximum or the optimum gain of uniformly spaced arrays was first investigated by Uzkov[4] in a very elegent formulation. By means of an orthogonal transformation in vector space, he obtained some important results concerning the optimum gain of an array of isotropic sources. In particular, he showed that the optimum gain of an end-fire array of isotropic sources as the separation approaches zero is numerically equal to N^2, where N denotes the number of sources. He also showed that when the separation is equal to $\lambda/2$, the optimum gain is numerically equal to N. Although he indicated that the method can be applied to arrays of directive sources, he did not elaborate. The excitation of the arrays to produce the optimum gain was not discussed in his work. Several years later, Block, Medhurst and Pool[5] proposed an optimization method based upon the impedance matrix defined for the elements of an array. Only a very limited calculation was reported in their work. Recently, Stearn[6] made some extensive calculations on the gain of an end-fire array of half-wave dipoles, and its corresponding excitation for various separations based upon their method.

In this work, we shall formulate the problem with the aid of an array matrix. The method, in certain respects, is equivalent to the impedance method except that all the essential results, namely, the gain and the corresponding excitations, can be expressed directly in terms of the elements of an array matrix. The formulation is particularly effective in dealing with arrays of short dipoles as well as isotropic sources. In the present report, we shall consider only broadside arrays, and ordinary end-fire arrays. The latter are characterized by a restraint

that the progressive phase shift between adjacent elements is assumed to be equal to the electrical distance between them. This type of end-fire array does not yield the optimum gain, because of the restraint on the phase shift. We treat these arrays here because the formulation is very much like that for broadside arrays. In a subsequent report, a general treatment of end-fire arrays will be given without imposing the above-mentioned restraint on the progressive phase shift.

FORMULATION

As an illustration of the method, the simplest case of a broadside array consisting of 2n isotropic sources as shown in Fig. 1(a) will be treated here. The field pattern of such an array is given by

$$(1) \qquad F(\theta) = \sum_{i=1}^{n} A_i \cos\left[\left(i - \frac{1}{2}\right) D \cos \theta\right]$$

where

$$D = 2\pi d/\lambda$$

d = separation between adjacent elements

θ = angle of inclination measured from the axis of the array

A_i = amplitude of the i-th pair of elements

$2n = N$ = number of elements.

The amplitude A_i is assumed to be real, but it may be either positive or negative. The corresponding power pattern can then be written in the form

$$(2) \qquad S(\theta) = [F(\theta)]^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} A_i A_j \cos\left[\left(i - \frac{1}{2}\right) D \cos \theta\right] \cdot$$

$$\cos\left[\left(j - \frac{1}{2}\right) D \cos \theta\right] .$$

The directivity or the gain of the array, designed to be broadside, is determined by

$$(3) \qquad g_N = \frac{S\left(\frac{\pi}{2}\right)}{\frac{1}{4\pi} \int \int S(\theta) \, d\Omega} = \frac{S\left(\frac{\pi}{2}\right)}{\frac{1}{2} \int_0^\pi S(\theta) \sin\theta \, d\theta}$$

$$= \frac{\left(\sum\limits_{i=1}^n A_i\right)^2}{\sum\limits_{i=1}^n \sum\limits_{j=1}^n \beta_{ij} A_i A_j}$$

where

$$(4) \qquad \beta_{ij} = \beta_{ji} = \frac{1}{2} \int_0^\pi \cos\left[\left(i - \frac{1}{2}\right)D\cos\theta\right]\cos\left[\left(j - \frac{1}{2}D\right)\cos\theta\right]\sin\theta \, d\theta$$

$$= \frac{1}{2}\left[s_{i-j}(D) + s_{i+j-1}(D)\right].$$

The $s_r(D)$ function appearing in (4) is defined by

$$(5) \qquad \begin{cases} s_r(D) = \dfrac{\sin rD}{rD}\,, & r \neq 0; \\[2em] & \text{and} \\[1em] s_0(D) = 1\,, & r = 0. \end{cases}$$

To optimize the gain, we set

$$(6) \qquad \frac{\partial g_N}{\partial A_p} = 0, \qquad p = 1,\ 2,\ \cdots n$$

which yields the following set of equations:

(7)
$$\sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij} A_i A_j - \left(\sum_{i=1}^{n} A_i \right) \left(\sum_{j=1}^{n} \beta_{pj} A_j \right) = 0,$$

$$p = 1, \ 2, \ \cdots \ n;$$

or

(8)
$$\sum_{j=1}^{n} \beta_{pj} A_j = \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij} A_i A_j \bigg/ \sum_{i=1}^{n} A_i$$

$$p = 1, \ 2, \ \cdots \ n.$$

Since the quantity on the right side of (8) is independent of p, we may denote it by K, and (8) can be written, in the matrix form,

(9)
$$[\beta] \{A\} = \{K\} \ .$$

The elements of the square matrix $[\beta]$, which is symmetric, are given by (4). For convenience, $[\beta]$ will be designated as the array matrix. Denoting the adjoint of $[\beta]$ by $[B]$, one has

(10)
$$\{A\} = [\beta]^{-1} \{K\} = |\beta|^{-1} [B] \{K\}$$

where $|\beta|$ denotes the determinant of the array matrix. From (10), we conclude that for optimum gain, the amplitude distribution must satisfy the following relations:

(11)
$$A_1 : A_2 : \cdots A_n = \sum_{j=1}^{n} B_{1j} : \sum_{j=1}^{n} B_{2j} : \cdots \sum_{j=1}^{n} B_{nj} \ .$$

In view of (8), Eq. (3) may be written as:

$$(12) \qquad G_N = \sum_{i=1}^{n} A_i \bigg/ \sum_{j=1}^{n} \beta_{pj} A_j; \qquad p = 1, \text{ or } 2, \cdots \text{ or } n$$

where the capital letter G_N signifies the optimum value of g_N. By making use of (11), and a multiplication rule in determinant theory, (12) can be converted into the following expression:

$$(13) \qquad G_N = \sum_{i=1}^{n} \sum_{j=1}^{n} B_{ij} / |\beta| \ .$$

Equations (11) and (13) constitute the principal result of this formulation. The simplicity of these formulae is manifest since only the elements of the array matrix are contained in these expressions. For the case of a broadside array with an odd number of elements equal to 2n-1, Fig. 1(b), Eqs. (11) and (13), are still valid, except that the elements of the array matrix are changed to

$$(14) \qquad \beta_{ij} = \frac{1}{2} (s_{i-j} + s_{i+j-2}) , \qquad \text{for } N = 2n - 1.$$

The method described here can be applied equally well to end-fire arrays, or arrays of directive sources. In the following section we list the expression for the power pattern and the corresponding array matrix for several types of arrays composed of isotropic sources or of short dipoles. It is assumed that the individual short dipoles have a figure-eight pattern described by $\sin \theta_d$, where θ_d denotes the polar angle measured from the axis of the dipole. For completeness, the cases discussed previously are also included.

POWER PATTERN AND THE CORRESPONDING ARRAY
MATRIX FOR VARIOUS TYPES OF ARRAYS

(I) <u>Broadside array of isotropic sources</u> (Fig. 1)

a) $N = 2n$

$$S_e(\theta) = \left[\sum_{i=1}^{n} A_i \cos \left(i - \frac{1}{2} \right) D \cos \theta \right]^2$$

$$\beta_{ij} = \frac{1}{2} [s_{i-j}(D) + s_{i+j-1}(D)]$$

where

$$s_o(D) = 1, \qquad s_r(D) = \frac{\sin rD}{rD}, \qquad D = \frac{2\pi}{\lambda} d.$$

b) $N = 2n-1$

$$S_o(\theta) = \left[\sum_{i=1}^{n} A_i \cos(i-1)D \cos \theta \right]^2$$

$$\beta_{ij} = \frac{1}{2} [s_{i-j}(D) + s_{i+j-2}(D)].$$
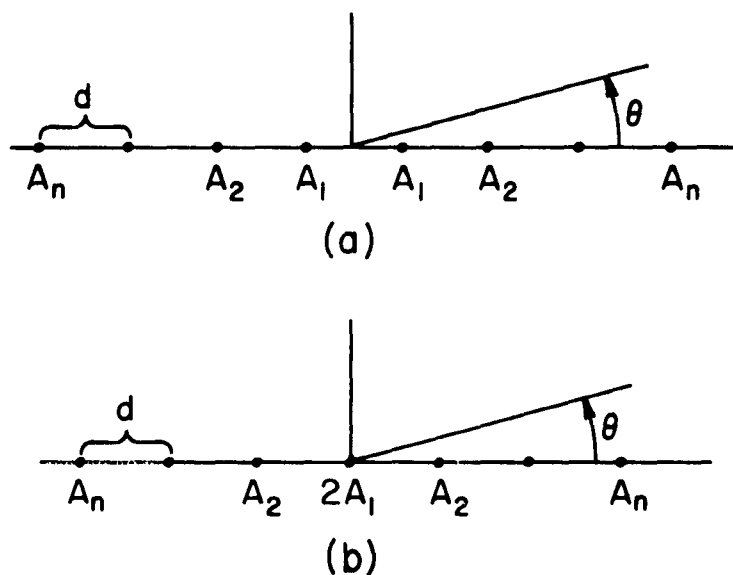
Fig. 1. Broadside arrays of isotropic sources (a) $N = 2n$, (b) $N = 2n-1$. The designation for the elements of an array with odd number of elements also applies to other types of arrays sketched in the subsequent figures.

In the following tabulation, the functions $S_e(\theta)$ and $S_o(\theta)$ have the same meaning as those defined for broadside arrays of isotropic sources.

(II) <u>Broadside array of parallel dipoles</u> (Fig. 2)

a) N = 2n

$$S(\theta, \phi) = (1 - \sin^2\theta \cos^2\phi)\, S_e(\theta)$$

$$\beta_{ij} = \frac{1}{2}\, [p_{i-j}(D) + p_{i+j-1}(D)]$$

where

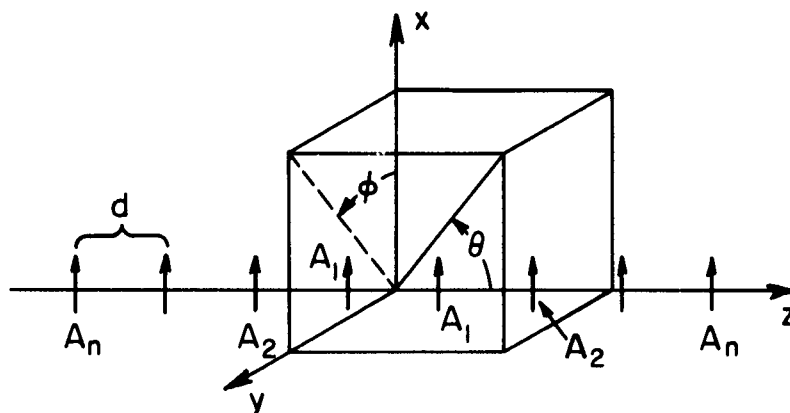$$p_0(D) = \frac{2}{3}, \quad p_r(D) = \left(1 - \frac{1}{r^2 D^2}\right) \frac{\sin rD}{rD} + \frac{\cos rD}{r^2 D^2}\ .$$

b) N = 2n - 1

$$S(\theta, \phi) = (1 - \sin^2\theta \cos^2\phi)\, S_o(\theta)$$

$$\beta_{ij} = \frac{1}{2}\, [p_{i-j}(D) + p_{i+j-2}(D)]\ .$$

Fig. 2. A broadside array of parallel dipoles.
The angles θ and ϕ appearing in the
text correspond to the ones shown in
this figure.

(III) Broadside array of collinear dipoles (Fig. 3)

a) $N = 2n$

$$S(\theta) = \sin^2 \theta \, S_e(\theta)$$

$$\beta_{ij} = \ell_{i-j}(D) + \ell_{i+j-1}(D)$$

where

$$\ell_o(D) = \frac{1}{3}, \quad \ell_r(D) = \frac{\sin \, rD}{r^3 \, D^3} - \frac{\cos \, rD}{r^2 \, D^2} \qquad .$$

b) $N = 2n - 1$

$$S(\theta) = \sin^2 \theta \, S_o(\theta)$$

$$\beta_{ij} = \ell_{i-j}(D) + \ell_{i+j-2}(D).$$
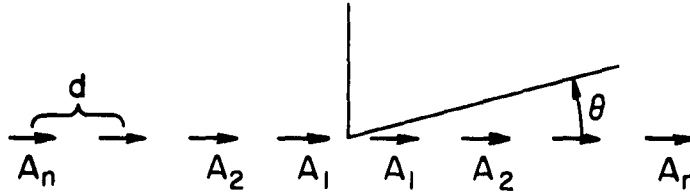
Fig. 3. A broadside array of collinear dipoles.

(IV) Broadside array of crossed dipoles (Fig. 4)

In this case the two crossed dipoles are assumed to be of equal amplitude but excited 90° out of phase. The field due to a single unit in the broadside direction ($\theta = \pi/2$, $\phi = \pi/2$) is therefore circularly polarized.

a) $N = 2n$

$$S(\theta, \phi) = \frac{1}{2} (1 + \sin^2 \phi \, \sin^2 \theta) \, S_e(\theta)$$

$$\beta_{ij} = \frac{1}{4} [C_{i-j}(D) + C_{i+j-1}(D)]$$

where

$$C_0(D) = \frac{4}{3}, \quad C_r(D) = \left(1 + \frac{1}{r^2 D^2}\right) \frac{\sin rD}{rD} - \frac{\cos rD}{r^2 D^2} \quad .$$

b) N = 2n - 1

$$S(\theta, \phi) = \frac{1}{2}(1 + \sin^2 \phi \, \sin^2 \theta)S_0(\theta)$$

$$\beta_{ij} = \frac{1}{4}[C_{i-j}(D) + C_{i+j-2}(D)].$$
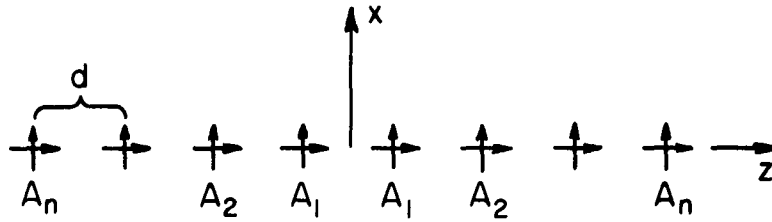
Fig. 4. A broadside array of crossed dipoles;
the excitation of the first pair is:
$A_{1x} = A_1$ and $A_{1z} = jA_1$, etc. The
angles θ and ϕ are shown in Fig. 2.

(V) <u>End-fire array of isotropic sources</u> (Fig. 5)

As explained in the introduction, the end-fire arrays considered
here are of the ordinary type, with a progressive phase shift between
adjacent elements numerically equal to D.

a) N = 2n

$$S_e'(\theta) = \left\{ \sum_{i=1}^{n} A_i \cos\left[\left(i - \frac{1}{2}\right)D(1 - \cos \theta)\right] \right\}^2$$

$$\beta_{ij} = \frac{1}{2}[s_{2(i-j)}(D) + s_{2(i+j-1)}(D)]$$

where the $s_r(D)$ function is the same as that defined in Case (I). We
may remark that $s_{2r}(D)$ can be decomposed as follows:

$$s_{2r}(D) = \cos rD \; \frac{\sin rD}{rD} = \cos rD \; s_r(D).$$

b) N = 2n-1

$$S_o'(\theta) = \left\{ \sum_{i=1}^{n} A_i \cos[(i-1)D(1-\cos \theta)] \right\}^2$$

$$\beta_{ij} = \frac{1}{2} \left[s_{2(i-j)}(D) + s_{2(i+j-2)}(D) \right].$$
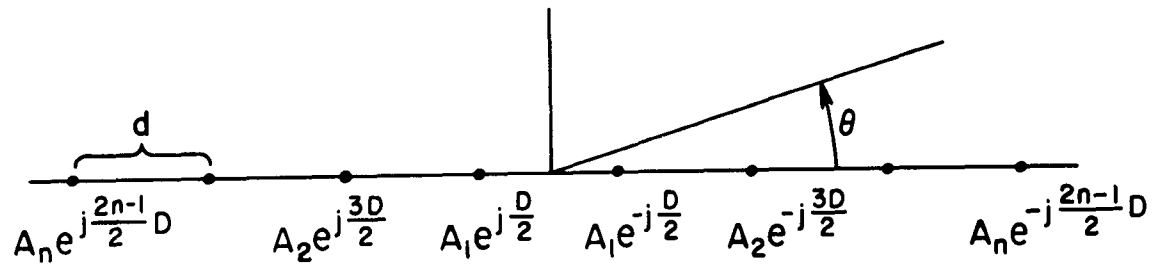
Fig. 5. An end-fire array of isotropic
sources, $D = 2\pi(d/\lambda)$.

(VI) **End-fire array of parallel dipole**   (Fig. 6)

a) N = 2n

$$S(\theta, \phi) = (1 - \sin^2 \theta \cos^2 \phi) \; S_e'(\theta)$$

$$\beta_{ij} = \frac{1}{2} \left[q_{i-j}(D) + q_{i+j-1}(D) \right]$$

where $S_e'(\theta)$ is the same as that defined in Case (V-a) and

$$q_r(D) = \cos rD \left[\left(1 - \frac{1}{r^2 D^2} \right) \frac{\sin rD}{rD} + \frac{\cos rD}{r^2 D^2} \right]$$

$$= \cos rD \; p_r(D).$$

The function $p_r(D)$ appears in Case (II).

b) $N = 2n-1$

$$S(\theta,\phi) = (1 - \sin^2\theta \cos^2\phi)\, S_0'(\theta)$$

$$\beta_{ij} = \frac{1}{2}\, [q_{i-j}(D) + q_{i+j-2}(D)]$$

where $S_0'(\theta)$ is defined in Case (V-b).
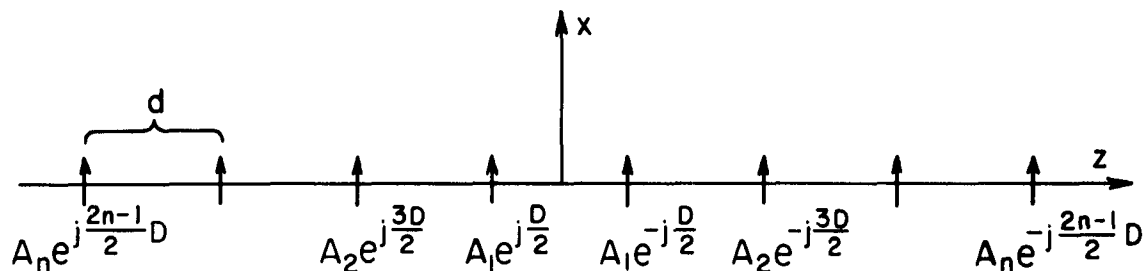
Fig. 6. An end-fire array of parallel dipoles;
the angles θ and ϕ are shown in Fig. 2.

(VII) <u>End-fire array of crossed dipoles</u>   (Fig. 7)

The crossed dipoles are assumed to be of equal amplitude but excited 90° out of phase.

a) $N = 2n$

$$S(\theta) = \frac{1}{2}\, (1 + \cos^2\theta)\, S_e'(\theta)$$

$$\beta_{ij} = \frac{1}{2}\, [q_{i-j} + q_{i+j-1}] \ .$$

Although the pattern for this case is different from that of an end-fire array of parallel dipoles, the array matrix, and hence the optimum gain, is the same for both cases.

b) $N = 2n-1$

$$S(\theta) = \frac{1}{2}(1 + \cos^2 \theta)\, S_o'(\theta)$$

$$\beta_{ij} = \frac{1}{2}\,[q_{i-j} + q_{i+j-2}] \ .$$
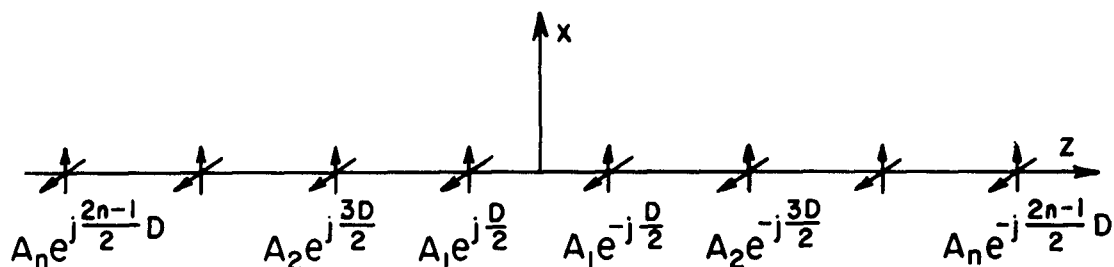
Fig. 7. An end-fire array of crossed dipoles; the excitation of the first unit at the right is:

$$A_{1x} = A_1\, e^{-j\frac{D}{2}}, \quad A_{1y} = jA_1\, e^{-j\frac{D}{2}}, \quad \text{etc.}$$

The angle θ and ϕ are shown in Fig. 2.

NUMERICAL COMPUTATION

The numerical computations for the amplitude distribution and the optimum gain based upon (11) and (13) seem to be straightforward. This is indeed the case if the order of the array matrix is equal to two, or if D, the separation between the elements, is not too small. When n is greater than two, and D is less than π for the broadside arrays or less than $\pi/2$ for the ordinary end-fire arrays, the computation becomes rather difficult even with the aid of an IBM-7090 computer. The fact that such a giant machine could not evaluate accurately a 3 x 3 determinant or invert the associated matrix without going into multiple precision programming is truly unexpected. It should be mentioned that $D = \pi$ for the broadside cases, and $D = \pi/2$ for the ordinary end-fire cases, correspond to the demarcations of the Dolph synthesis and the Riblet's extension. For convenience, we shall call the region of D lying below these two values the "super-gain" region. Pending a more accurate evaluation, we shall present here only the data of G_N for the values of D where the accuracy was certain. These results are plotted in Fig. 8-13. In regard to the corresponding amplitude distributions, the

data are too numerous. Figures 14-19 give a few sample curves for some typical distributions. In the case of broadside arrays of isotropic sources, as discussed in the following section, it is possible to find the limiting values of G_N as d → 0. The dotted lines of the G_N curves in the super-gain region correspond to the interpolated values between these limiting values and the accurately computed values. For other cases, where these limiting values have not yet been found, only the accurately computed values are plotted. Each set of curves contain also a plot of G_{max}/G_∞ and G_{min}/G_∞ where G_{max} and G_{min} denote, respectively, the first maximum and the first minimum of the G_N curves. G_∞ denotes the asymptotic value of G_N as d → ∞. Numerically, G_∞ is equal to N for arrays of isotropic sources, and is equal to 3/2 N for arrays of dipoles. In the case of broadside arrays of isotropic sources, we also plotted a curve, Fig. 8(e), labelled G_0/G_∞ where G_0 denotes the limiting value of G_N as d → 0. A discussion on the values of G_0 is given in the next section.

THE LIMITING VALUE OF G_N FOR BROADSIDE ARRAYS OF ISOTROPIC SOURCES

For N = 3 or 4, it is relatively simple to evaluate the limiting value of G_N as D approaches zero. This can be done by expanding the relevant function $s_n(D)$ in a series of D^2 and retaining the leading terms of

$$|\beta| \quad \text{and} \quad \sum \sum B_{ij}.$$

The result gives

(15) $$\lim_{D \to 0} G_3 = \lim_{D \to 0} G_4 = \left(\frac{3}{2}\right)^2 = 2.25.$$

When N = 5 or 6, the same procedure calls for seven terms in the series expansion of $s_n(D)$ which already involves a very tedious algebraic manipulation. The result yields

(16) $$\lim_{D \to 0} G_5 = \lim_{D \to 0} G_6 = \left(\frac{3}{2} \quad \frac{5}{4}\right)^2 = 3.515625 .$$

To obtain this simple result we must deal with many clumsy numbers such as $(6)^{14}/15!$. It should be mentioned here that these calculations were performed before we had access to Uzkov's article. In view of the orderly figures contained in (15) and (16), we then proposed a

conjecture that for arrays with odd number of elements N or those with even number of elements N + 1 the limiting value must be given by

$$(17) \qquad \lim_{D \to 0} G_N = \left(\frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{6} \cdots \frac{N}{N-1} \right)^2 .$$

After reading Uzkov's paper, we realized that as far as these limiting values are concerned they can most conveniently be obtained from his method. Although Uzkov gave these values only for end-fire arrays, it can be shown by means of the orthogonal transformation which he outlined that, in general,

$$(18) \qquad \lim_{D \to 0} G_N(\theta_0) = \sum_{n=0}^{N-1} (2n+1) \, P_n^2 \, (\cos \theta_0)$$

where $P_n(\cos \theta_0)$ denotes the Legendre polynomial of order n, and θ_0 denotes the angle, measured with respect to the axis of the array, in which direction the array was designed for the optimum gain. The values of

$$\lim_{D \to 0} G_n(\theta_0)$$

are plotted, on db scale, in Fig. 20(a) and (b). For end-fire arrays, without the phase restraint mentioned previously in the introduction, one has

$$(19) \qquad \lim_{D \to 0} G_N(0) = \sum_{n=0}^{N-1} (2n+1) = N^2 .$$

This formula was originally enunciated by Uzkov. For the broadside array, (18) reduces to

$$(20) \qquad \lim_{D \to 0} G_N \left(\frac{\pi}{2} \right) = \sum_{n=0}^{N-1} (2n+1) \, P_n^2 (1).$$

The same formula applies to the ordinary end-fire arrays as $D \to 0$.
It can be shown that the sum given by (20) is indeed identical to the one
given in product form by (17). Looking back at the hard work which we
did in arriving at (17), one has to admire Uzkov's ingenious method of
successive orthogonal transformations in providing the answer for these
limiting values of G_N. The only consolation for our innocent hard work
is that without it, the alternative expression for (20) as given by (17)
probably would not be recognized at first glance. In regard to the
numerical values of the limiting values of $G_N(\pi/2)$ as given by (17) or
(20), it is of some interest to point out that these can be approximated
quite accurately by an even simpler formula. From the Peirce-Foster
Table,[7] one finds that

$$(21) \qquad \frac{4n}{\pi} > \left(\frac{3}{2} \cdot \frac{5}{4} \cdots \frac{2n-1}{2n-2} \right)^2 > \frac{2(2n-1)}{\pi} \quad ;$$

hence the mean value of $4n/\pi$ and $\left(4n-2/\pi \right)$ may be used an an approxi-
mate value of $G_N(\pi/2)$, i.e.,

$$(22) \qquad G_N\left(\frac{\pi}{2} \right) \doteq \frac{2(2n-\frac{1}{2})}{\pi} = \frac{2N+1}{\pi} \quad .$$

The expression given in (22) is certainly the asymptotic value of
$G_N(\pi/2)$ for large values of N. The fact that it may be used for <u>all</u>
values of N in approximating (17) is clear from the following table:

TABLE OF $\left(\frac{3}{2} \cdot \frac{5}{4} \cdots \frac{N}{N-1} \right)^2$ AND $\frac{2N+1}{\pi}$

N	$\left(\dfrac{3}{2} \cdot \dfrac{5}{4} \cdots \dfrac{N}{2N-1} \right)^2$	$\dfrac{2N+1}{\pi}$
3	2.250	2.228
5	3.515	3.501
7	4.785	4.775
9	6.056	6.048
11	7.328	7.321
13	8.600	8.594
15	9.873	9.868
17	11.14	11.14
19	12.42	12.42

Finally, we wish to emphasize that the result found here also agrees with the conclusion drawn by Bouwkamp and Bruijn[8] that for a continuously distributed line source, $N \rightarrow \infty$, the optimum gain is without bound. However, for discrete arrays there is a unique solution.

COMPARISON OF G_N WITH THE GAIN OF A UNIFORMLY EXCITED ARRAY

Before we conclude this report, it is important to point out that the gain of a uniformly excited array with the same number of elements and spacing, which will be denoted by $g_N^{(u)}$, is, in general, slightly less than the optimum gain except in the super-gain region. Figures 21-25 show the comparison between $g_N^{(u)}$ and G_N for some typical cases. In view of the practical value of uniformly excited arrays, we have compiled a fairly complete set of curves of $g_N^{(u)}$ for various types of arrays. The data will soon be published as a separate report.

ACKNOWLEDGEMENT
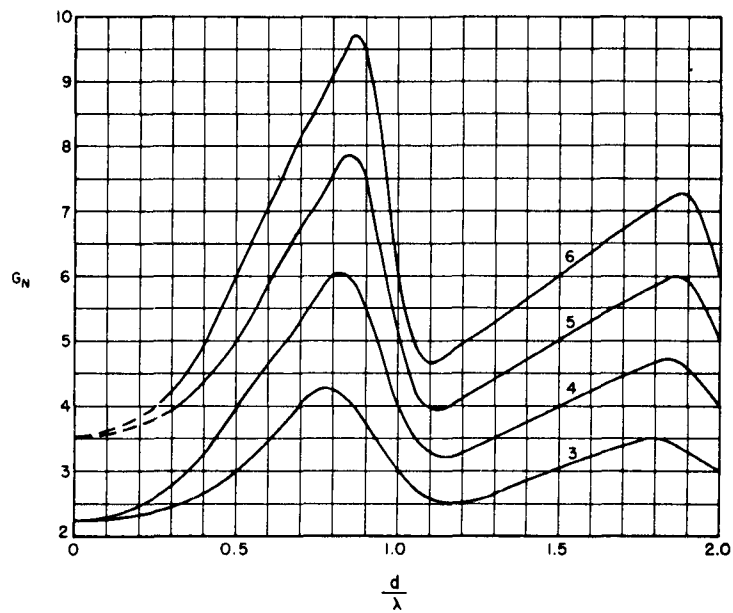
(a) N = 3-6
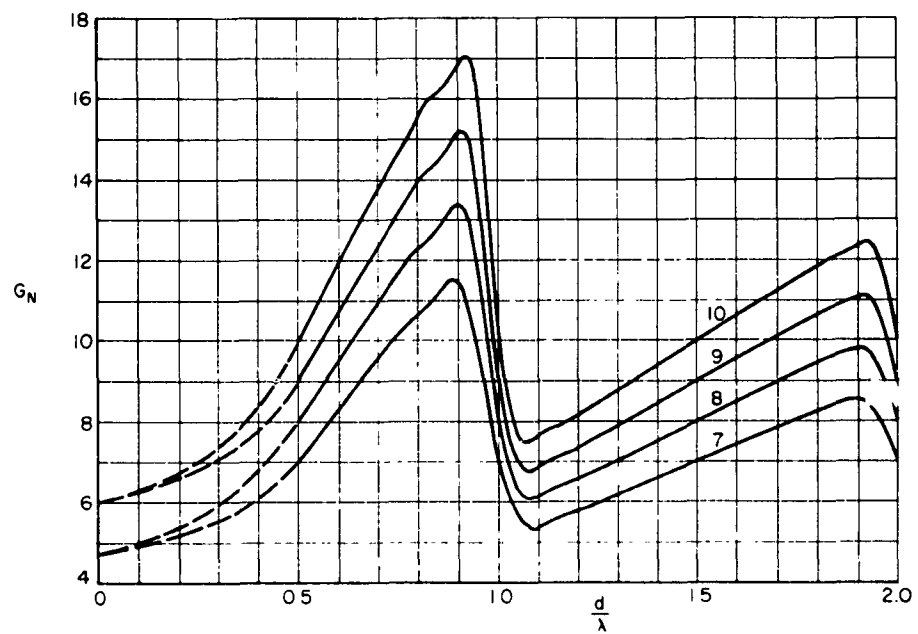
(b) N = 7-10

Fig. 8. Optimum gain of broadside arrays of isotropic sources.
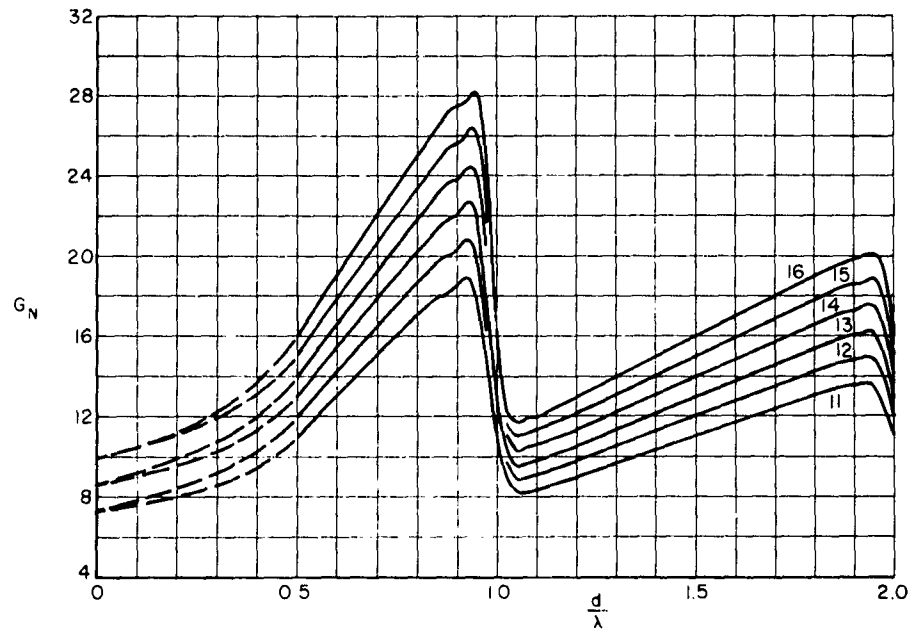
(c) N = 11-16
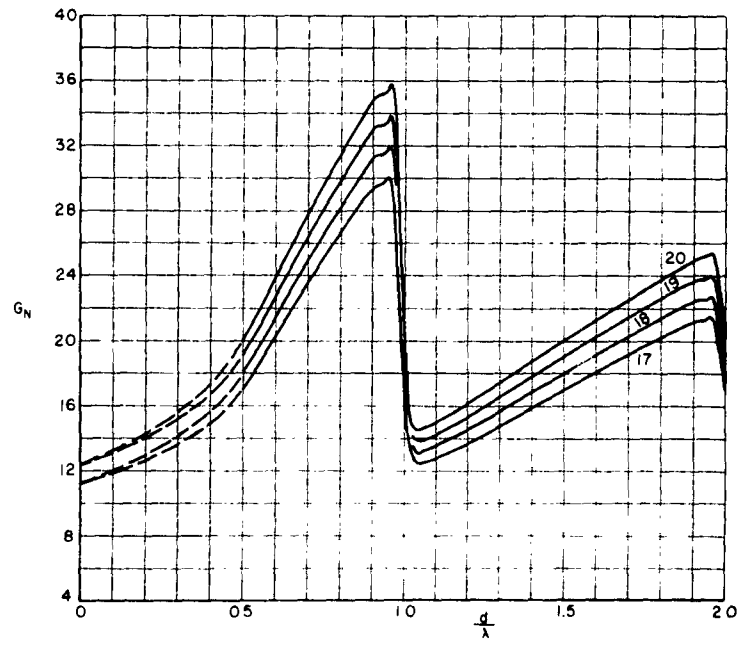
(d) N = 17-20
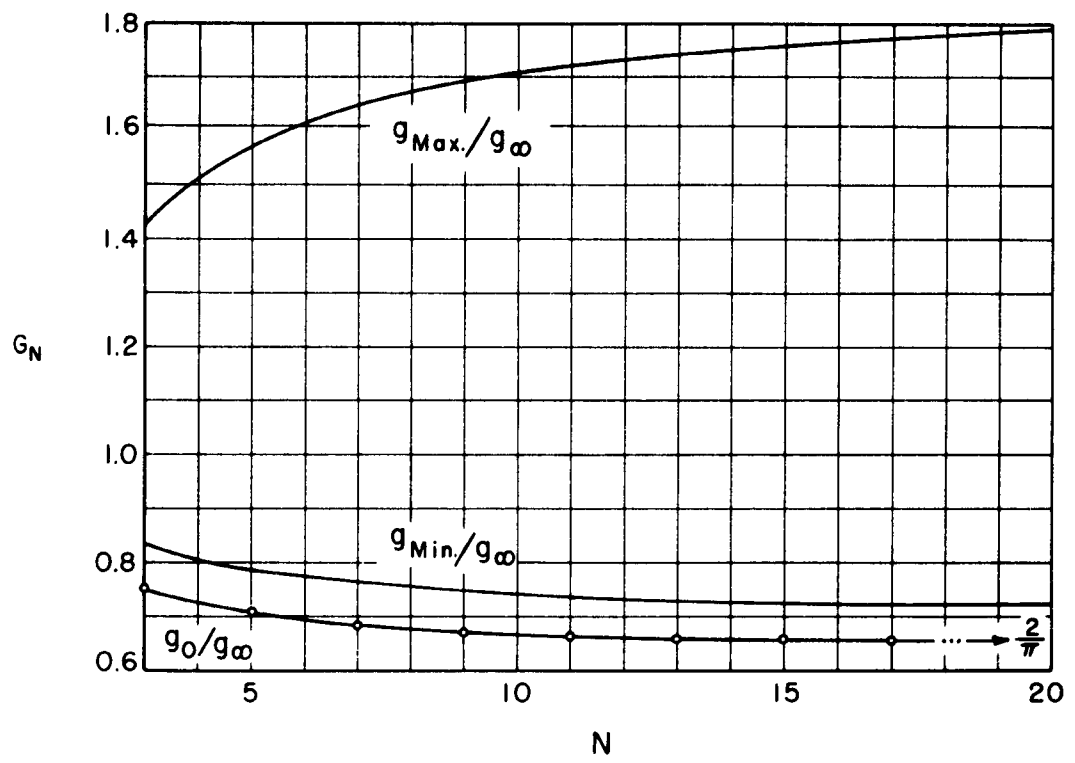
(e) g_{max}/g_∞, g_{min}/g_∞, and g_0/g_∞
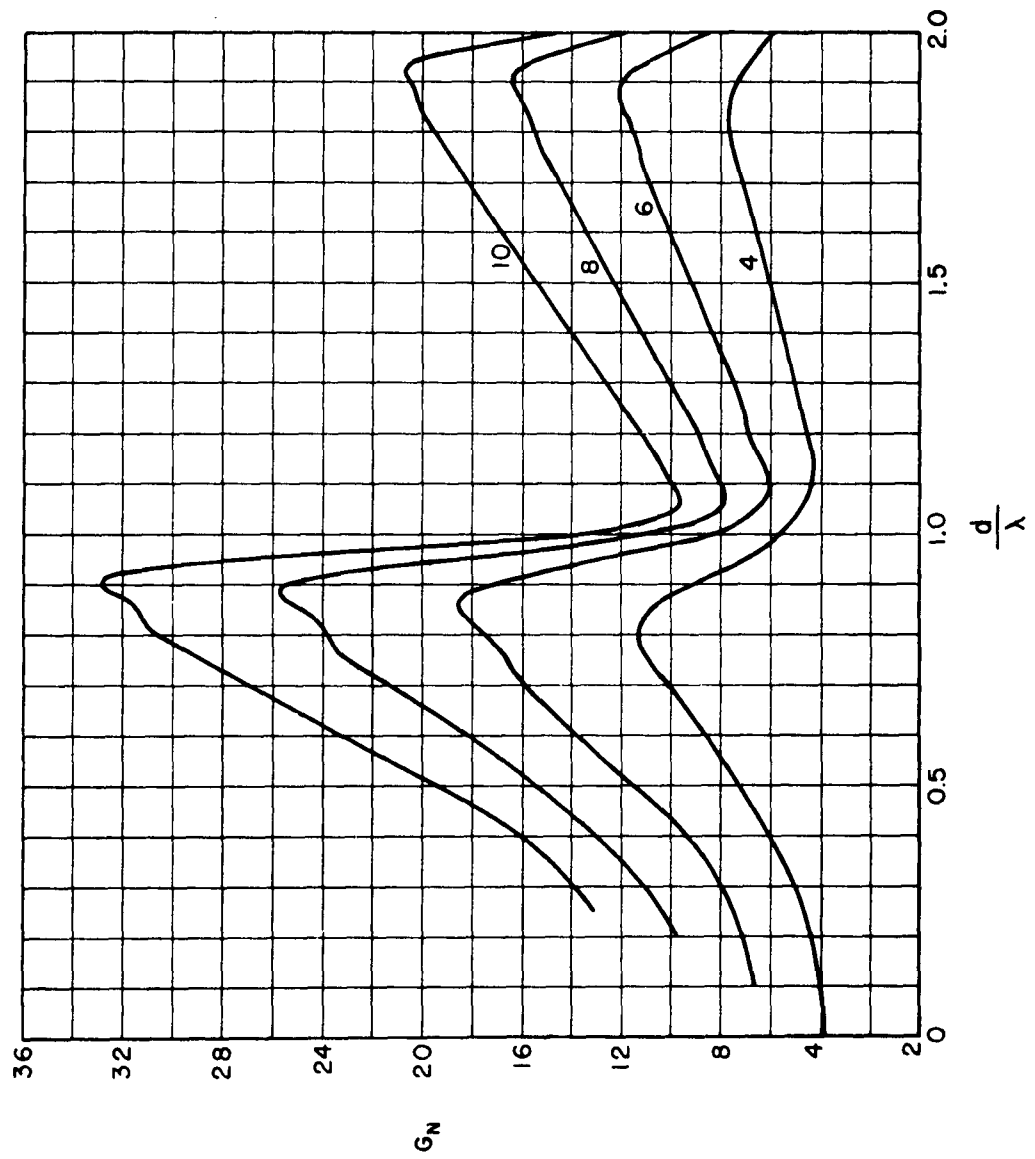
Fig. 9. Optimum gain of broadside arrays of parallel dipoles.
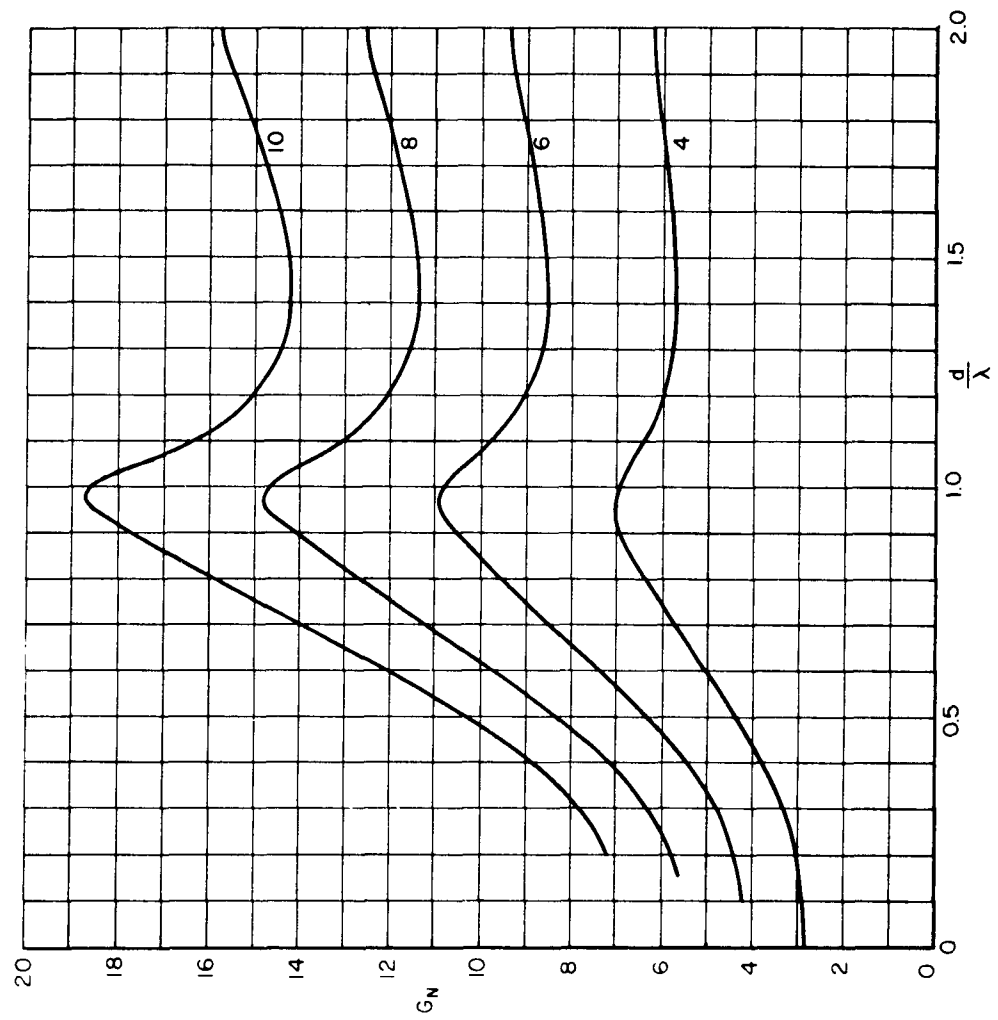
Fig. 10. Optimum gain of broadside arrays of collinear dipoles.
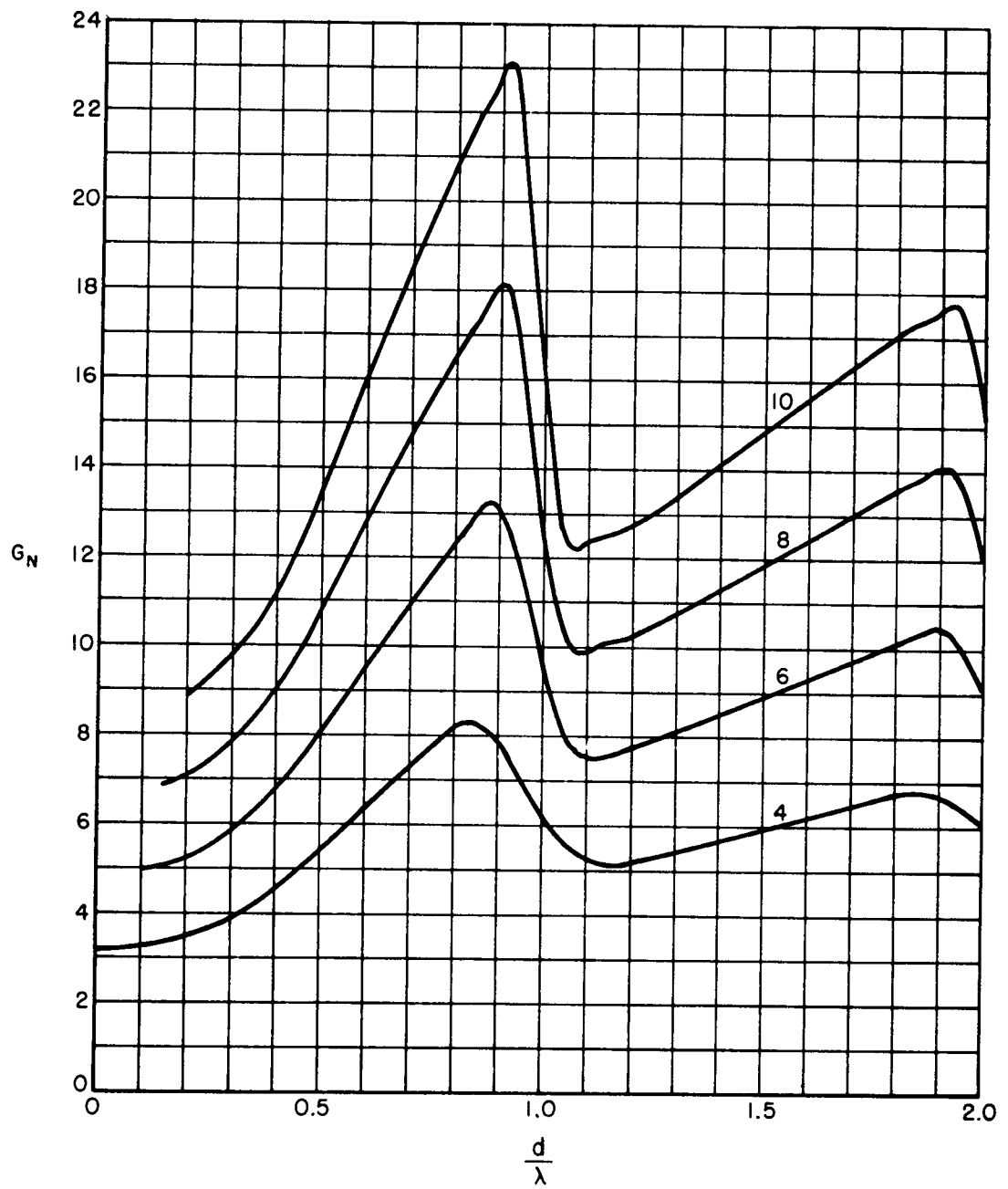
Fig. 11. Optimum gain of broadside arrays of crossed dipoles.
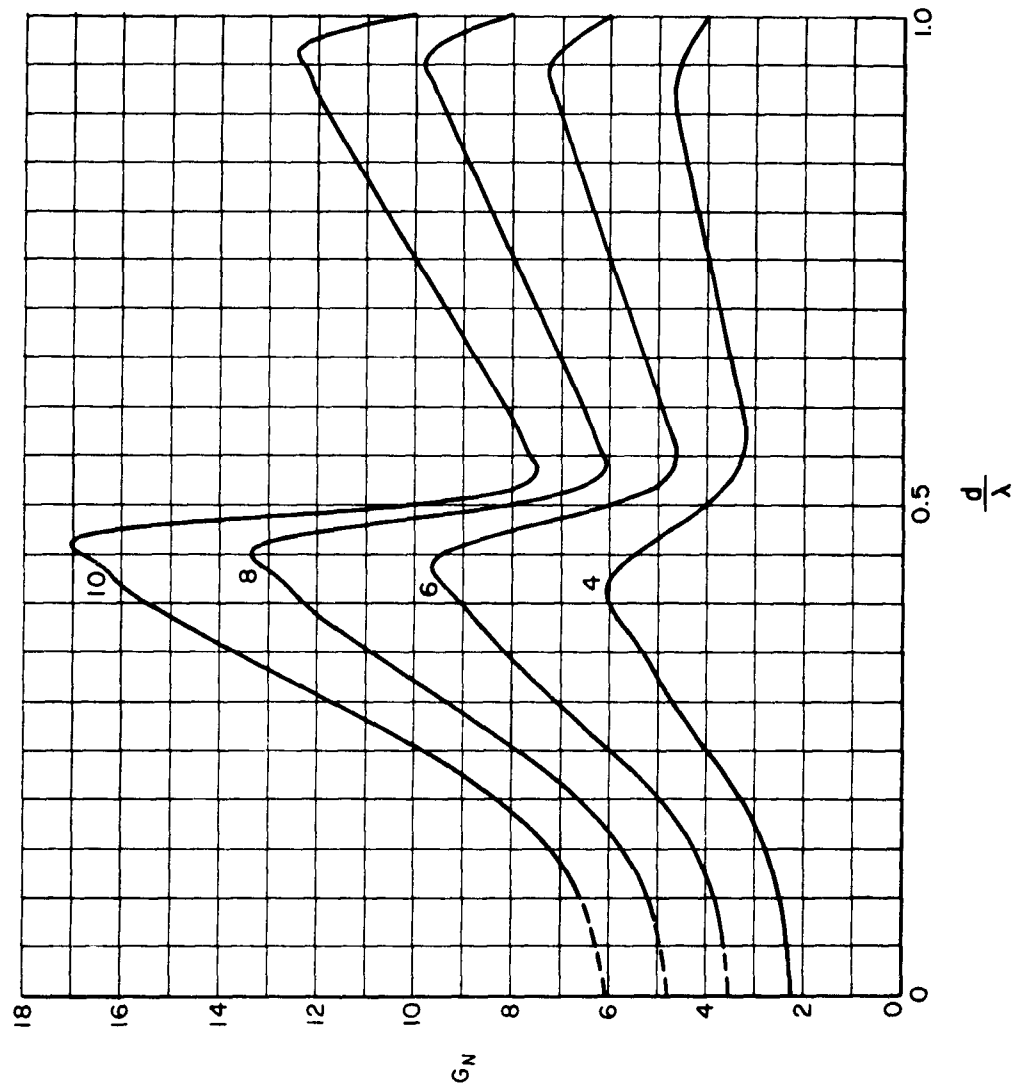
Fig. 12. Optimum gain of ordinary end-fire arrays of isotropic sources.
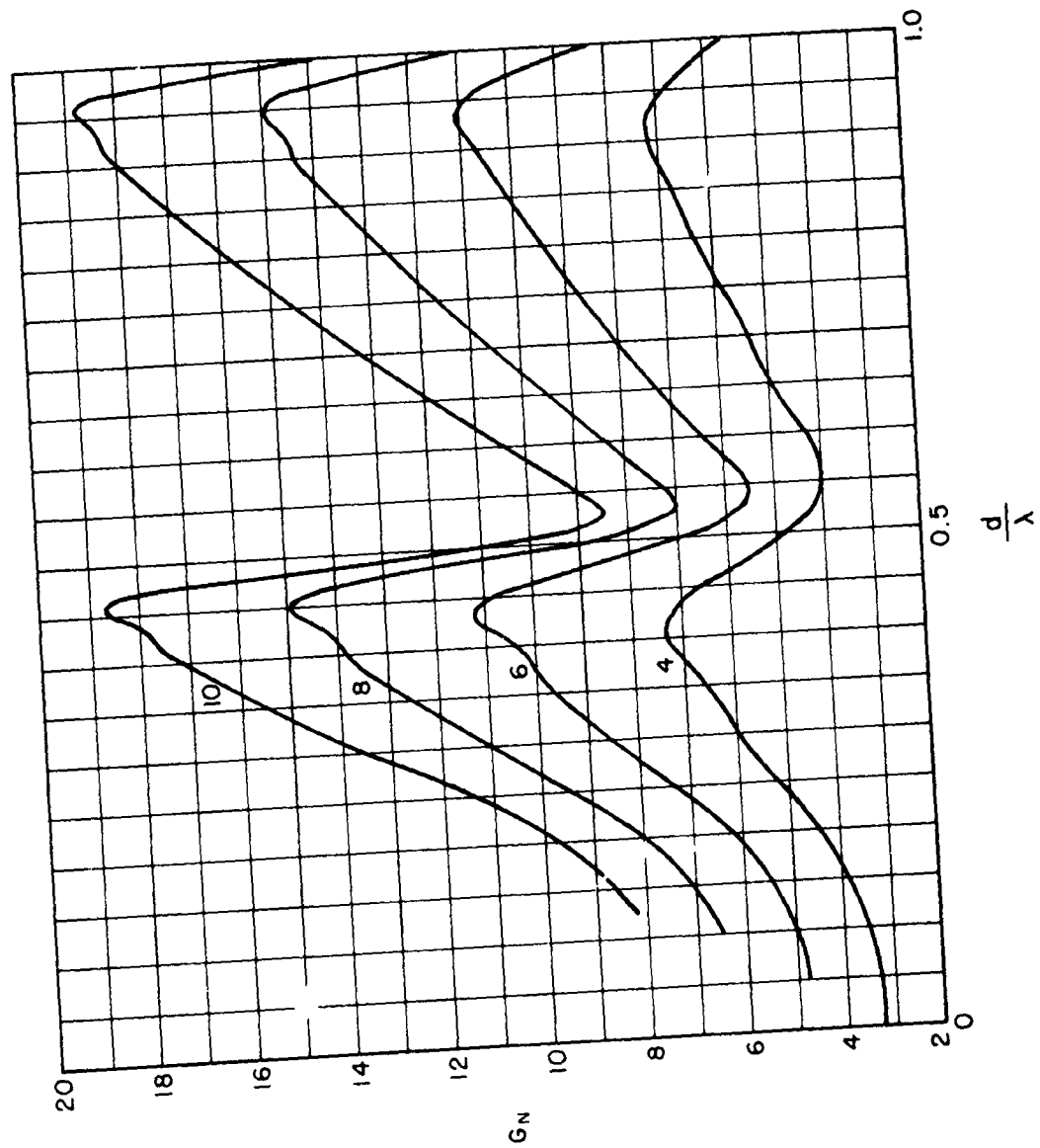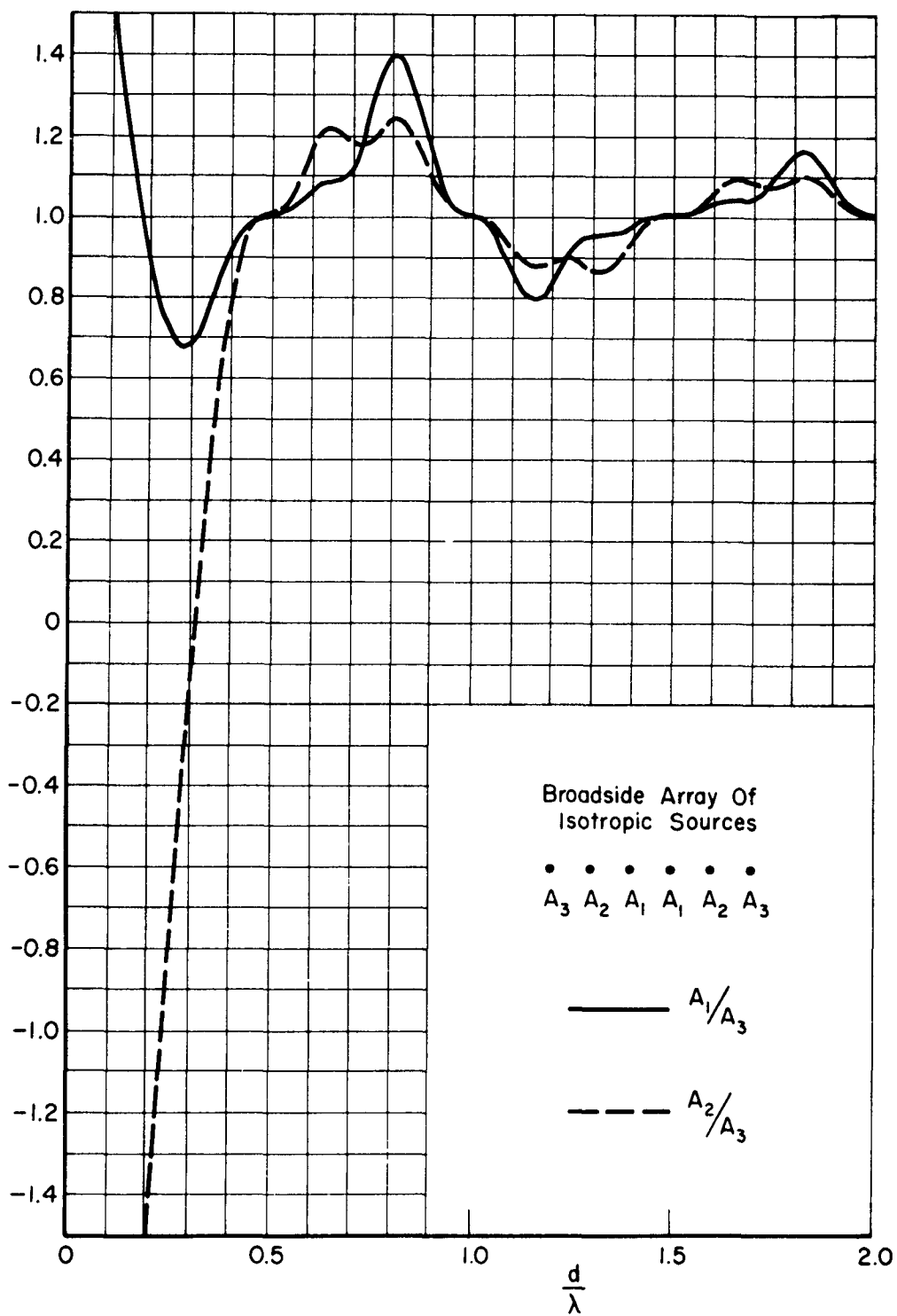
Fig. 13. Optimum gain of ordinary end-fire arrays of parallel dipoles.

24

Fig. 14. Amplitude distribution of a broadside array
of six isotropic sources.
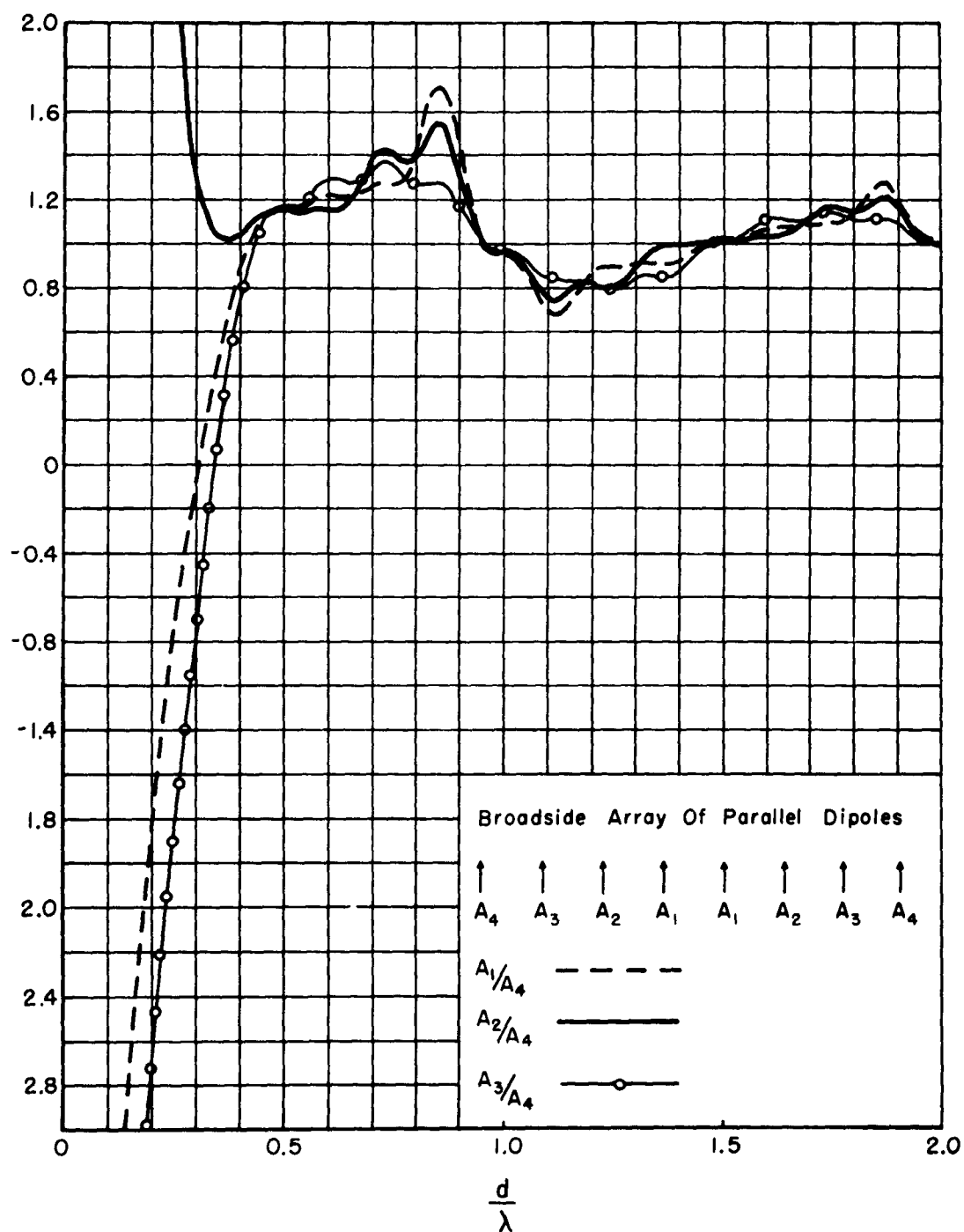
Fig. 15. Amplitude distribution of a broadside array
of eight parallel dipoles.
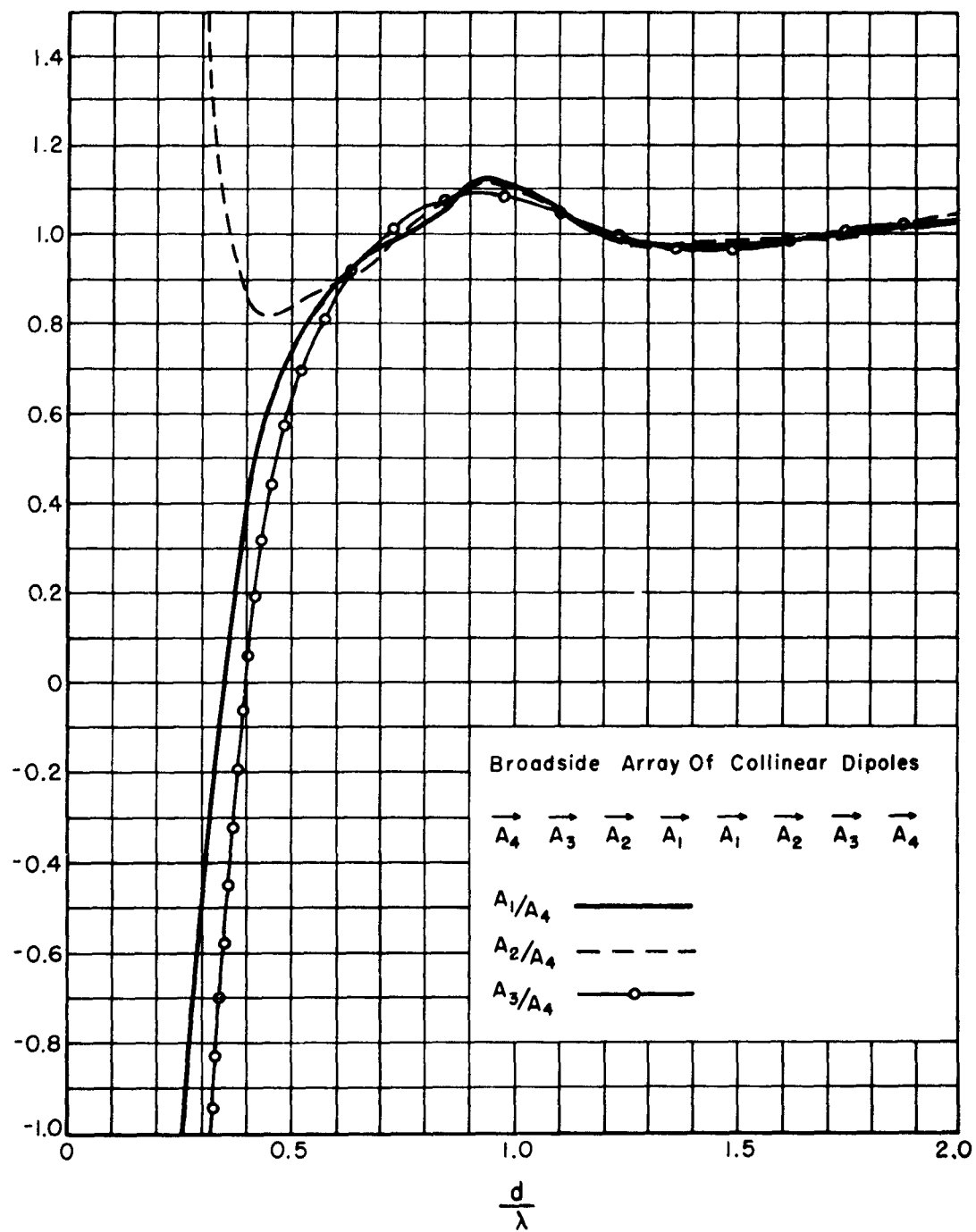
Fig. 16. Amplitude distribution of a broadside array of eight collinear dipoles.
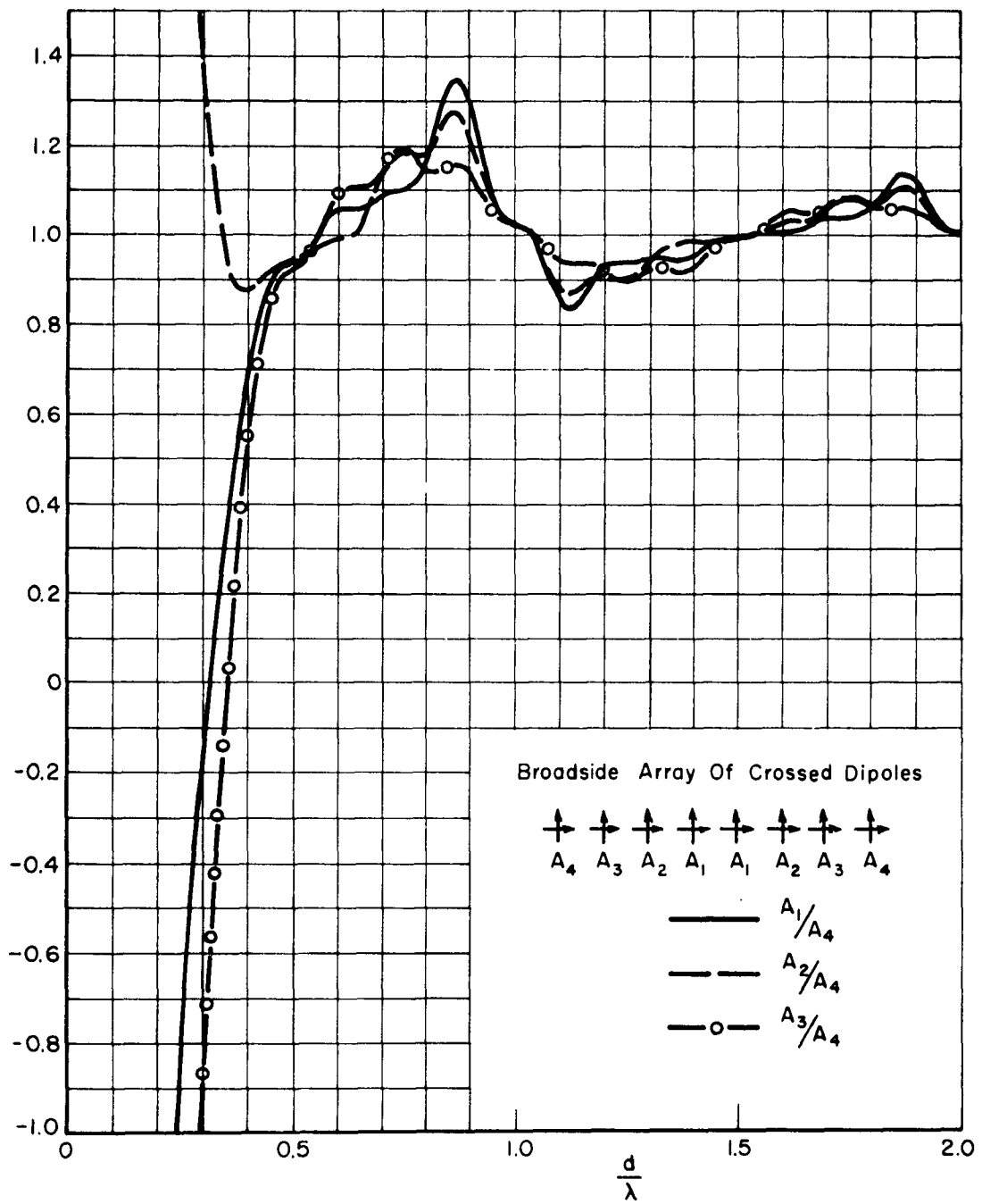
Fig. 17. Amplitude distribution of a broadside array
of eight crossed dipoles.
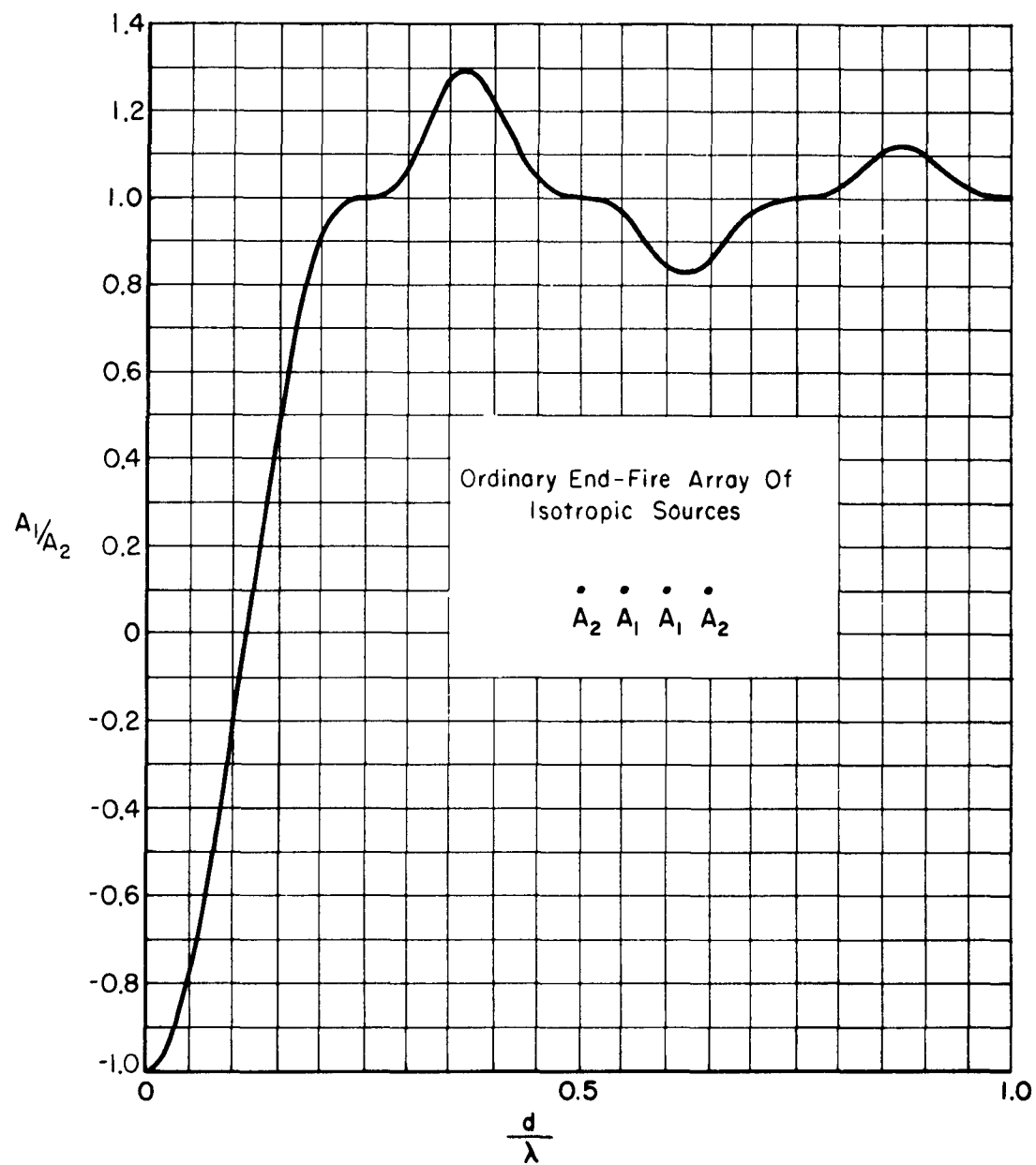
Fig. 18. Amplitude distribution of an ordinary end-fire
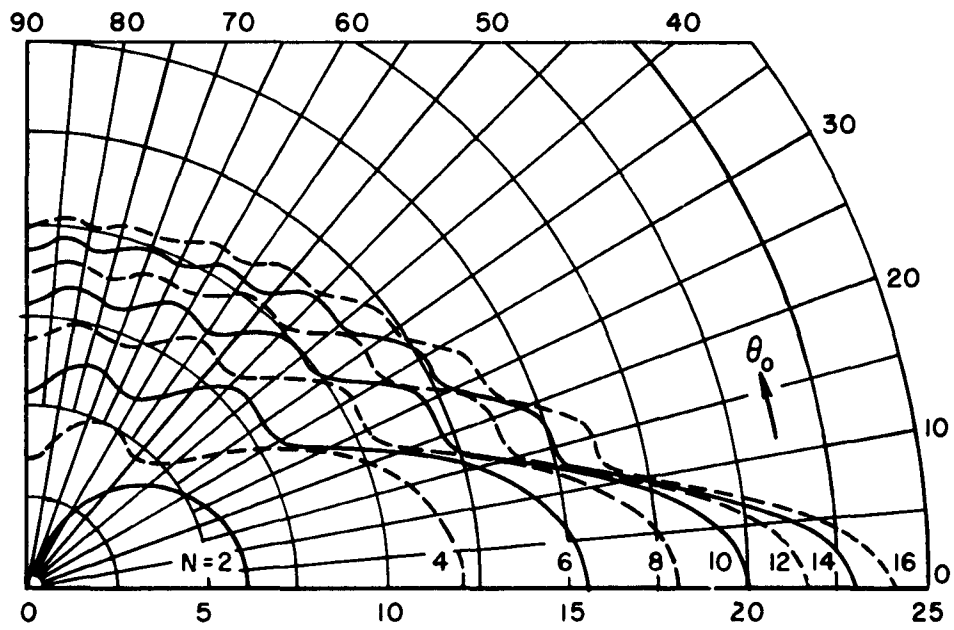array of four isotropic sources.
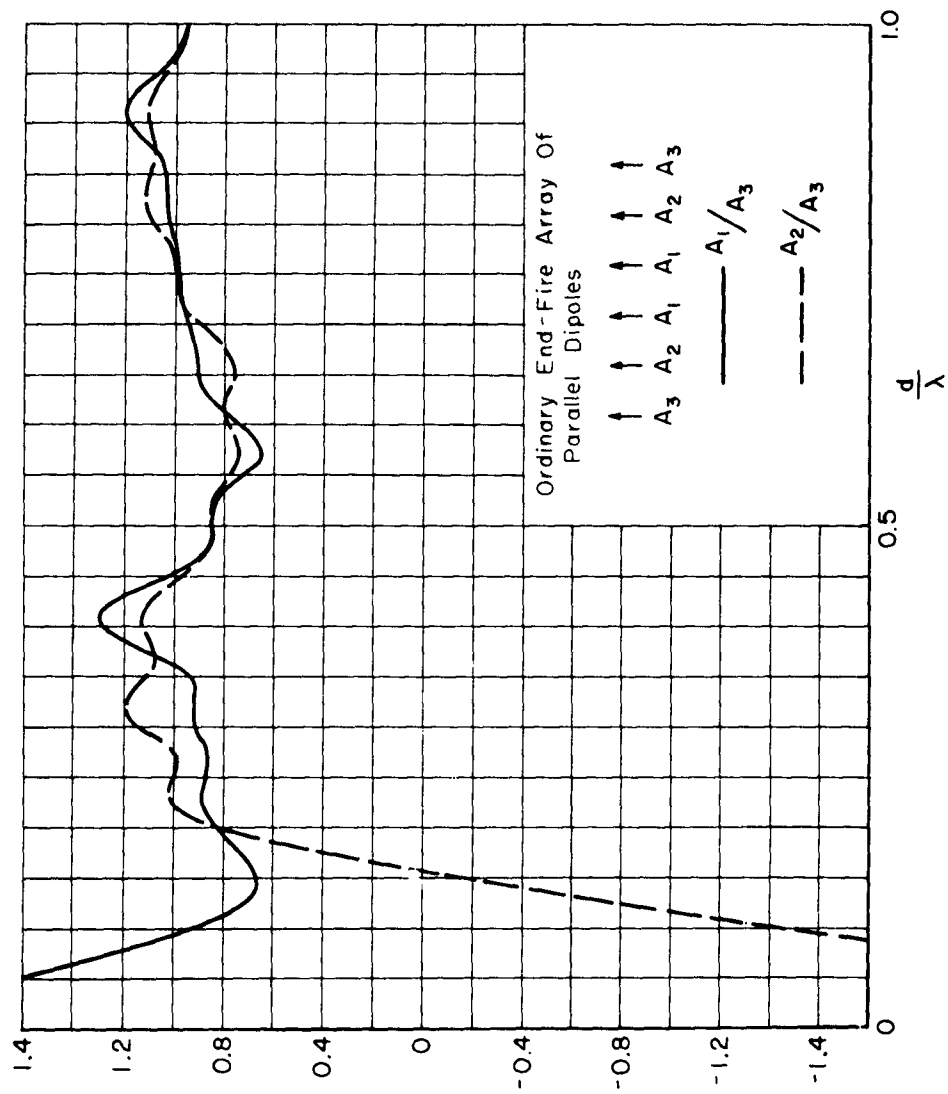
Fig. 19. Amplitude distribution of an ordinary end-fire array of six parallel dipoles.
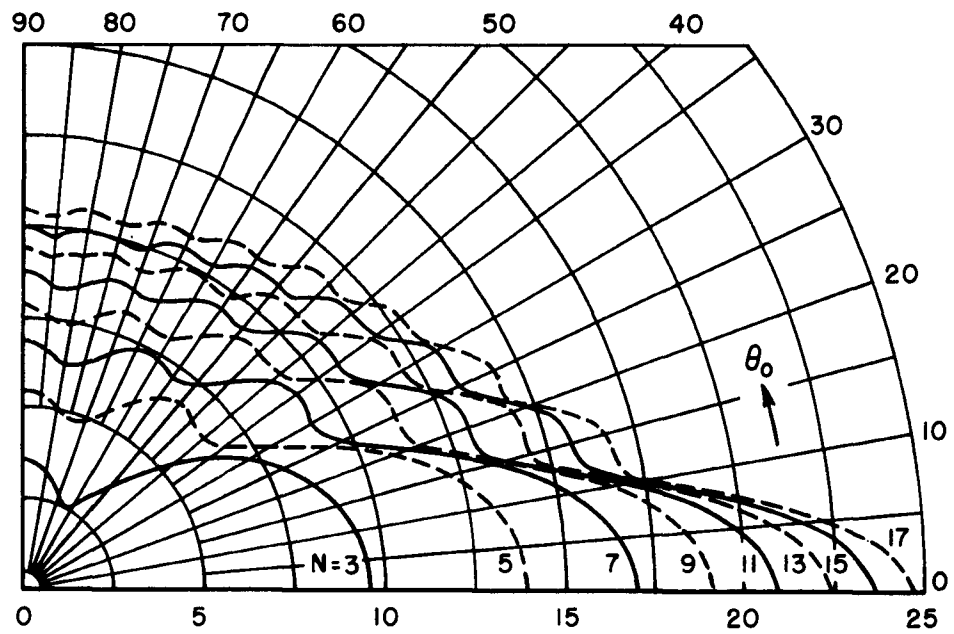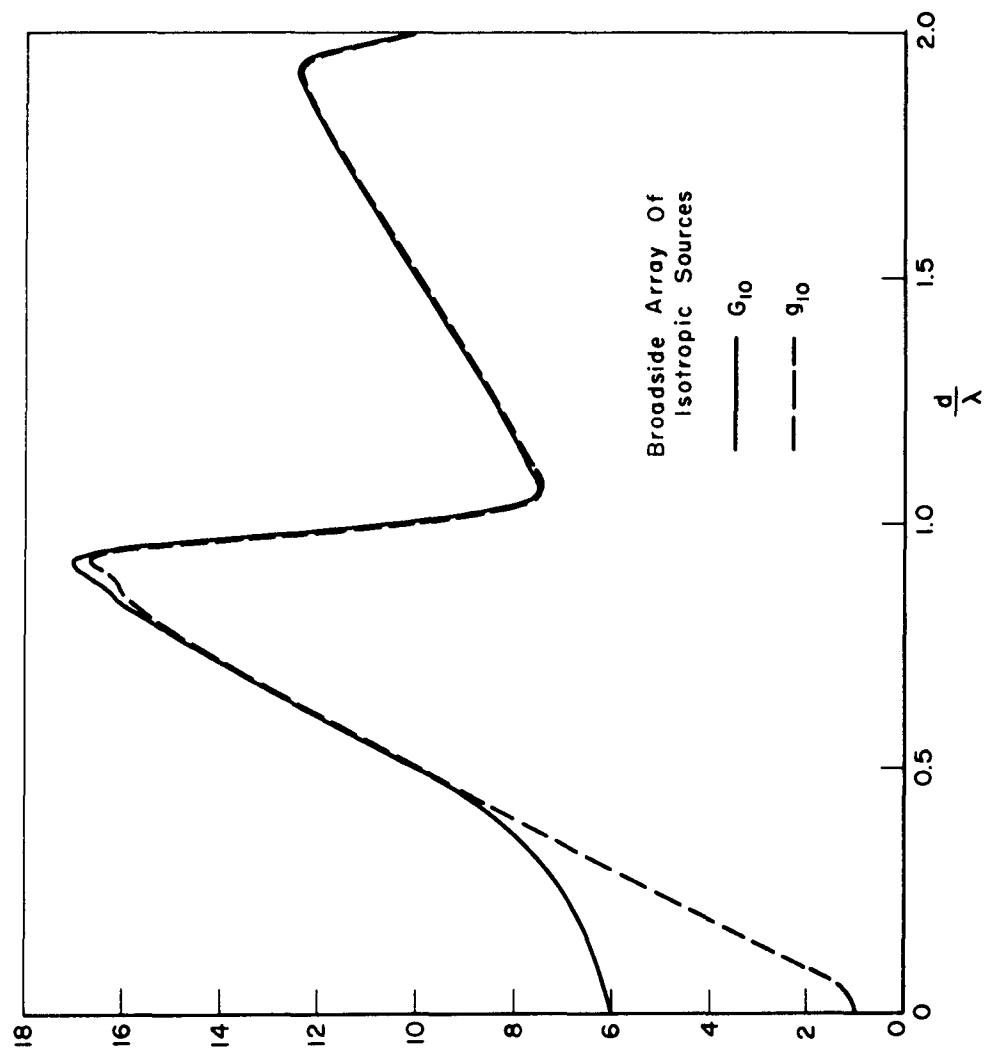
Fig. 20. Polar plot of $10 \log\lfloor \mathrm{Lim}\ G_N(\theta_o)\rfloor$,
$$D \to 0$$
(a) N = even,   (b) N = odd.

Fig. 21. Comparison of g_{10} and G_{10} of broadside arrays of isotropic sources.
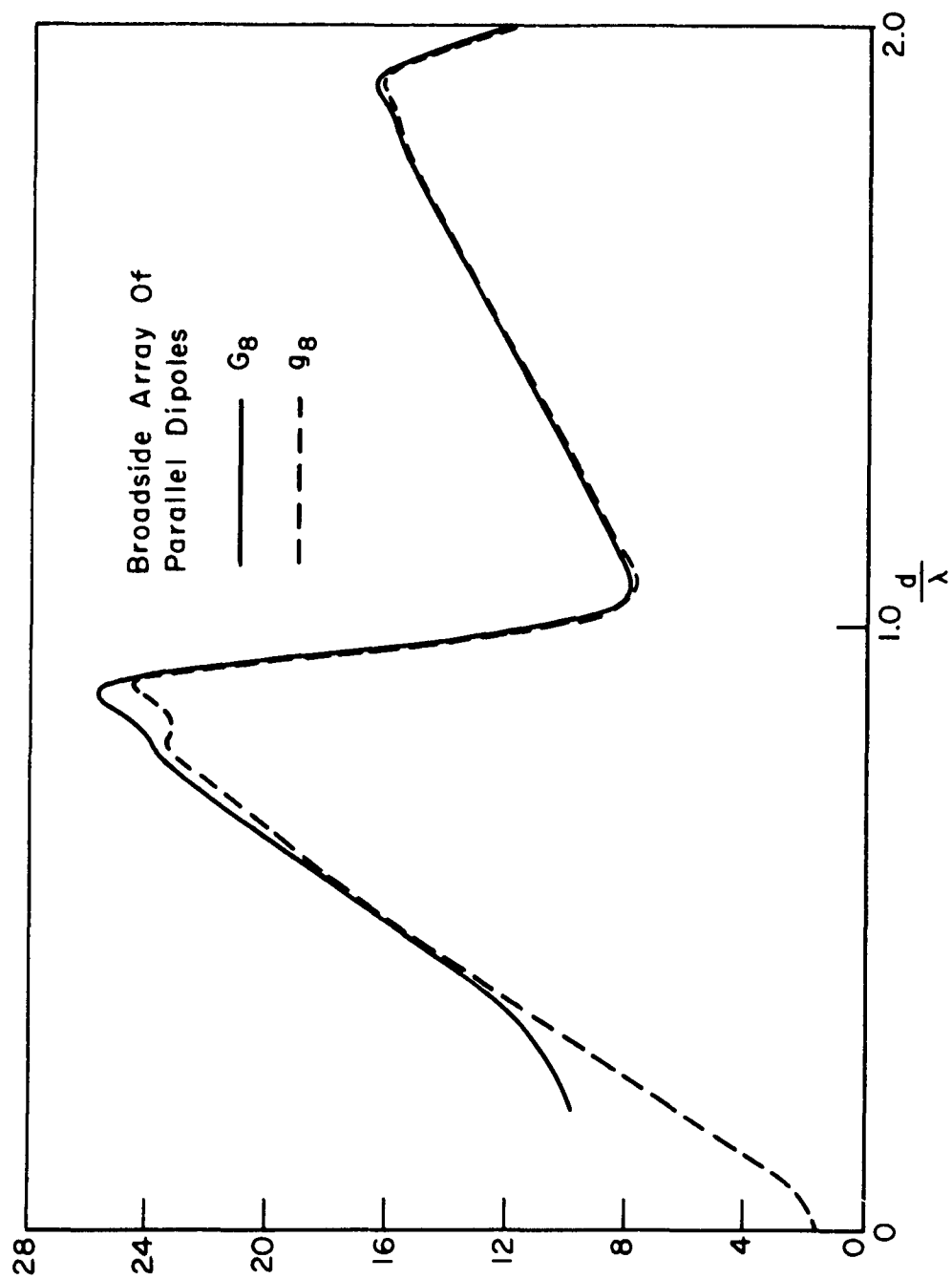
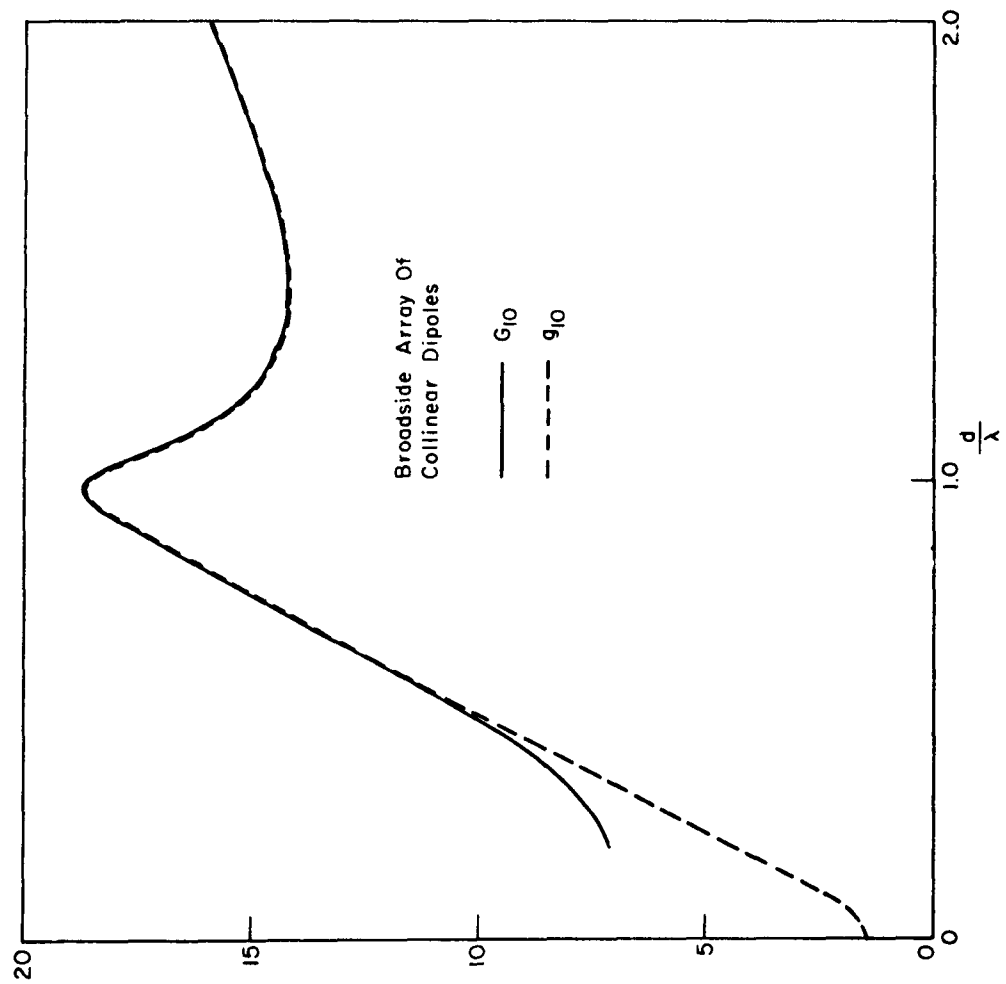Fig. 22. Comparison of $g_8^{(u)}$ and G_8 of broadside arrays of parallel dipoles.

Fig. 23. Comparison of $g_{10}^{(u)}$ and G_{10} of broadside arrays of collinear dipoles.
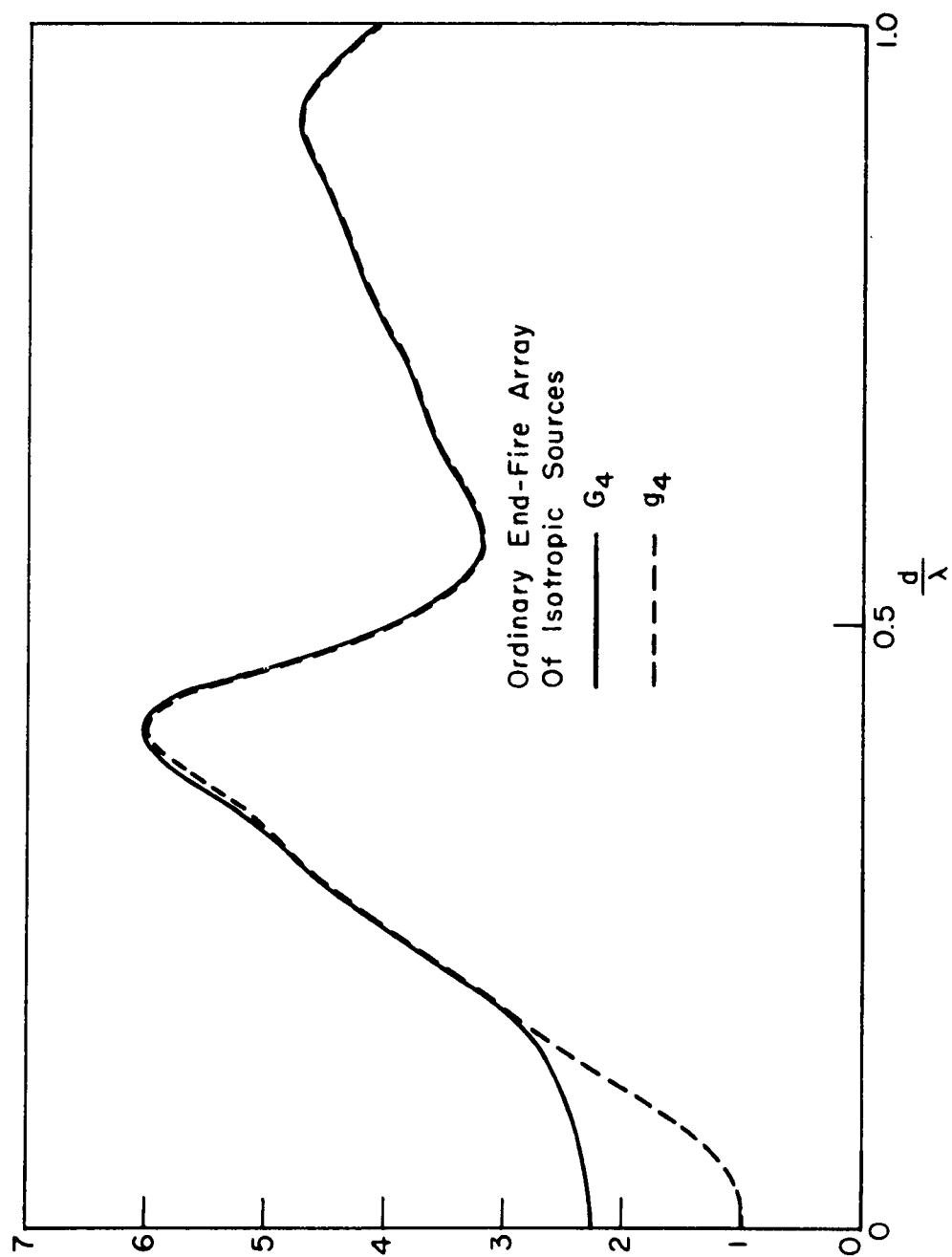
Fig. 24. Comparison of $g_4^{(u)}$ and G_4 of ordinary end-fire arrays of isotropic sources.
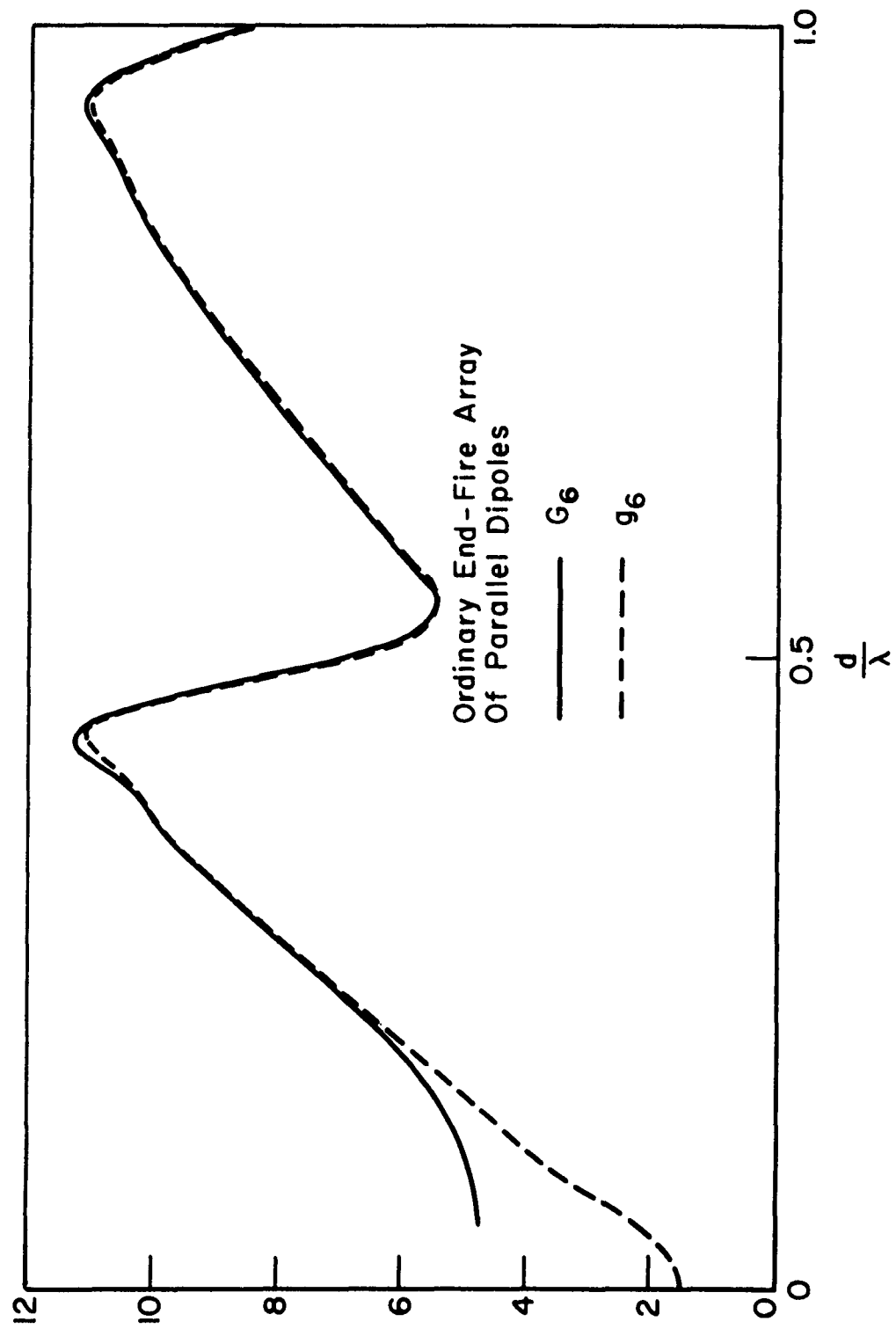
Fig. 25. Comparison of $g_6^{(u)}$ and G_6 of ordinary end-fire arrays of parallel dipoles.

REFERENCES

1. C.L. Dolph, "A current distribution for broadside arrays which optimizes the relationship between beam width and side-lobe level," Proc. I.R.E. Vol. 34, p. 335 (June 1946).

2. R.J. Riblet, "Discussion of Dolph's paper," Proc. I.R.E. Vol. 35, p. 489 (May 1947).

3. M.T. Ma and D.K. Cheng, "A critical study of linear arrays with equal side lobes," I.R.E. Convention Record, Part I, p. 110 (1961).

4. A.I. Uzkov, "An approach to the problem of optimum directive antenna design," Comptes Rendus (Doklady) de l'Academie de Sciences des l'URSS, Vol. LIII, No. 1, p. 35 (1946).

5. A. Bloch, R.G. Medhurst and S.D. Pool, "A new approach to the design of super-directive aerial arrays," J.I.E.E., 100, Part III, p. 303 (1953).

6. C.O. Stearns, "Computed performance of moderate size, super-gain antennas," Report No. 6797, Boulder Laboratories, National Bureau of Standards, U.S. Department of Commerce (September 1961).

7. B.O. Peirce and R.M. Foster, "A short Table of Integrals," Ginn and Company, Boston, p. 107 (1956).

8. C.J. Bouwkamp and N.G. deBruijn, "The problems of optimum antenna current distribution," Philips Research Report 1, p. 135 (1946).